Twin primes and Siegel Zeros

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31 août 2019

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Many thanks to Mr.Buì for this internship, during which I had the opportunity to learn a new aspect -that I knew nothing about- of number theory. This definitely confirms my love for analytic number theory.

1 Introduction

Every theorem or definition that a student in first year of master degree might not know should be defined/detailed in the appendix.

1.1 Generalities about sieve theory

The idea of *sieve* was introduced by Eratosthenes, 300 year before J-C. It was forgotten for a long time, but a Norwegian mathematician -Viggo Brun- gave a second birth to this theory by proving the following theorem :

Theorem 1.1.1 (Brun, 1919). *Let* $\mathscr{J}$ *be the set of twin primes. Then the following series is convergent*

$$\sum_{p\in\mathscr{J}} \frac{1}{p}$$

He then improved his theory for 7 years. Other mathematicians, like Selberg, improved the theory by adding new sieves. Some of these sieves were almost powerful enough to give a proof of the oldest and hardest conjectures, like the twin primes conjecture. For example, the Selberg Sieve, very versatile and general, can lead us to the following lower bound :

Theorem 1.1.2 (Selberg, 1950). *For* $n > 1$*, we write* $\Omega(n) := \sum_{p|n} \nu_p(n)$*. Then, we have the following result :*

$$\frac{x}{\log(x)^2} \ll_x \#\left\{x < n < 2x, \Omega(n) \leq 2 \text{ and } \Omega(n+2) \leq 3\right\}$$

This method, relatively easy to understand, still manages to lead us to a strong result, the strongest ever published on that subject at that moment. Chen Jingrun then slightly improved the result, at the price of a way more complex proof and more fecund ideas :

Theorem 1.1.3 (Chen, 1966). *There are infinitely many prime numbers* p *such that* $\Omega(p+2) \leq 2$.

Sieve theory seems to be what we can call *a miracle*, for it is incredibly powerful and allow mathematicians to prove result one didn't hope to prove before a long time. However, as one can see with the last two theorems, some of the obtained results are not "sharp" (by sharp we mean that we have to change deal with "almost prime" number) enough. This lack of sharpness could come from a phenomenon called *the parity problem*, which is no yet totally understood. We'll talk about it later, since the proof of Heath-Brown is based on a "trick" that waive the parity problem.

1.1.1 Example of sieve

In this section we'll talk about Selberg's sieve, since it's a tool used in the proof and not too hard to prove/use. The proof of what follows can be found in [Unk10].

Let's define some quantities before stating the main result. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of non-negative real numbers. We assume that $\sum a_n$ is convergent and we call the value of the sum of the series A. We also introduce α, a multiplicative function satisfying $0 \leq \alpha(p) < 1$ for all prime p. For each square-free d, we write

$$A_d := \sum_{m\in\mathbb{Z}} a_{md} = A\alpha(d) + r(d)$$

Let $\mathscr{P}$ be a finite set of primes, and let's define $P := \prod_{p\in\mathscr{P}} p$. What we want to estimate is the quantity $A(P)$, defined by

$$A(P) := \sum_{(n,P)=1} a_n$$

Finally, we write $\omega(n) := \sum_{p|n} 1$. It's now time to give Selberg's theorem.

Theorem 1.1.4. *For each* $z \geq 1$*, we have*

$$A(P) \leq \frac{A}{S(P,z)} + R(P,z)$$

where

$$S(P,z) := \sum_{d|P, d\leq z} \prod_{p|d} \frac{\alpha(p)}{1-\alpha(p)}, \quad R(P,z) := \sum_{d|P, d\leq z^2} 3^{\omega(d)} |r(d)|$$

The proof, based on the beautiful optimization of a quadratic form, uses squared sums of weights. It allowed Selberg to have a huge liberty of choice for the weights. We'll first need two lemmas.

Lemma 1.1.5. *Let* $Q := \sum_{i,j|P} \alpha([i,j])\lambda_i\lambda_j$*, a quadratic form in the* λ_d*. Then its minimum is* $1/S(P,z)$*, and it's attained for* λ_d *such that* $|\lambda_d| \leq 1$*.*

Proof. *Since for any non-negative integers* k, l*, we have* $[k,l](k,l) = kl$*, one can write* Q *as*

$$Q = \sum_{i,j|P} \frac{\alpha(i)\lambda_i\alpha(j)\lambda_j}{\alpha((i,j))}$$

That quadratic form is diagonalized by the introduction of a family $(\delta(e))_{e|P}$ *with* $1/\alpha(d) = \sum_{e|d} \delta(e)$*. Indeed, we now have*

$$Q = \sum_{e|P} \delta(e) \left(\sum_{e|d} \alpha(d)\lambda_d \right)^2$$

Let $x(e)$ *be defined by*

$$x(e) := \sum_{e|d} \lambda_d \alpha(d)$$

A Möbius inversion gives us

$$\delta(e) = \prod_{p|e} \frac{1-\alpha(p)}{\alpha(p)}, \quad \lambda_d = \frac{1}{\alpha(d)} \sum_{d|e} \mu(e/d)x(e)$$

To minimize the quadratic form, the conditions on the $x(e)$ *are that* $e > z \Rightarrow x(e) = 0$ *and* $\sum_{e|P} \mu(e)x(e) = \alpha(1)\lambda_1 = 1$*. By Cauchy-Schwarz inequality, the minimum of* Q *is*

$$\left(\sum_{e|P, e\leq z} \frac{1}{\delta(e)} \right)^{-1} = \frac{1}{S(P,z)}$$

It is attained at $x(e) = \mu(e)/(\delta(e), S(P,z))$*. We then deduce that*

$$S(P,z)\lambda_d = \frac{\mu(d)}{\alpha(d)} \sum_{d|e\leq z} \frac{1}{\delta(e)} = \frac{\mu(d)}{\alpha(d)\delta(d)} \sum_{f|(P,d), f\leq z/d} \frac{1}{\delta(f)}$$

But $1/(\alpha(d)\delta(d)) = \sum_{e|d} \delta(e)^{-1}$*, which implies that*

$$S(P,z)\lambda_d = \mu(d) \sum_e \sum_f \frac{1}{\delta(ef)}$$

where each $ef \leq z$ *and with no* ef *values repeated.*

The absolute value of the sum is lower than $S(P,z)$*, so we do have* $|\lambda_d| \leq 1$*.*

The second lemma concerns a result of combinatory.

Lemma 1.1.6. *Let d be a square-free integer. Then*

$$\sum_{[a,b]=d} 1 = 3^{\omega(d)}$$

Proof. *Let's write $d = p_1...p_r$. d is the lcm of a and b if and only if a is a divisor of d and b can be written as $(d/a) \times c$, where $c|a$.*

Let's count. For each divisor a of d, there are $2^{\omega(a)}$ divisors of a. So, for each divisor a of d, there are exactly $2^{\omega(a)}$ integers b with $[a,b] = d$. Then, there are $\sum_{k=0}^{\omega(d)} \binom{\omega(d)}{k} 2^k = 3^{\omega(d)}$ pairs of integers (a,b) such that $[a,b] = d$.

Let's now prove the theorem.

Proof. *Let $(\lambda_d)_{d|P}$ be arbitrary real numbers, with $\lambda_1 = 1$ and eventually $\lambda_d = 0$ once $d > z$. By consequence*

$$\begin{aligned} A(P) \leq \sum_n a_n \left(\sum_{d|(n,P)} \lambda_d \right)^2 &= \sum_{i,j|P} \lambda_i \lambda_j \sum_{[i,j]|n} a_n \\ &= \sum_{i,j|P} \lambda_i \lambda_j (A\alpha([i,j]) + r([i,j])) \\ &\leq AQ + R \end{aligned}$$

where R an error term, defined as follows

$$R := \sum_{i,j|P} |\lambda_i \lambda_j r([i,j])|$$

Thanks to the lemmas, it's easy to conclude.

To illustrate the power of this method, let's recall the beautiful theorem of Hadamard and De la Vallée Poussin

Theorem 1.1.7 (Hadamard-De la Vallée Poussin, 1896)**.**

$$\pi(x) \sim \frac{x}{\ln(x)}$$

When an Eratosthenes sieve would give us the "weak" bound (see [G.T08])

$$\pi(x) \leq (e^{-\gamma} + o(1)) \frac{x}{\ln_2 x}$$

the Selberg sieve gives us a bound of the right order

$$\pi(x) \ll \frac{x}{\ln x}$$

Proof. *Let x be an integer. We set $a_n = \chi_{[1,x]}(n)$. We fix $z > 1$ and $P = \prod_{p \leq z} p$. For d square-free, we have*

$$A_d = x/d + \underbrace{(\lfloor x/d \rfloor - x/d)}_{:=r(d)}$$

One can remark that $\pi(x) - \pi(z) \leq A(P)$ so that, by Selberg theorem (where we slightly modified the error term, taking in account the fact that $|r(d)| \leq 1$)

$$\pi(x) \leq \pi(z) + \frac{x}{S(P,z)} + O\left(\sum_{i|P, i<z} |\lambda_i| \right)^2$$

Of course, the error term is $O(z^2)$. Let's lower bound S. We have

$$S(P,z) = \sum_{d|P, d\leq z} \frac{1}{\phi(d)} \geq \sum_{d\leq z} \frac{\mu^2(d)}{\phi(d)}$$

$$\geq \sum_{d\leq z} d^{-1} - \sum_{\substack{d\leq z \\ d\ not \\ square-free}} d^{-1}$$

The left term is of order $\log z + O(1)$, and for the right one we have that

$$\sum_{\substack{d\leq z \\ d\ not \\ square-free}} d^{-1} \leq \frac{1}{4} \sum_{d\leq z/4} d^{-1}$$

Putting these together, we get that $\log z \ll S(P,z)$. Choosing $z = (x/\log(x))^{1/2}$, one easily concludes.

1.1.2 Parity Problem

A lot of what I learned about the parity problem is due to T.Tao, through his *What's new* blog (see in particular [Tao07]). As one may have noted, I almost (Selberg's estimate being the only exception) only gave upper-bound statements when talking about sieve theory. The reason is quite simple : getting a lower bound with sieve theory is usually a lot harder than simply giving an upper bound. It might come from what is known as the *parity problem*. The parity problem, identified and named by Selberg in 1949, is "roughly" defined by Tao :

> *If A is a set whose elements are all products of an odd number of primes (or are all products of an even number of primes), then (without injecting additional ingredients), sieve theory is unable to provide non-trivial lower bounds on the size of A. Also, any upper bounds must be off from the truth by a factor of 2 or more.*

Let's remind Chen's theorem (here called Theorem 1.1.3). It says that there are infinitely primes p such that $p+2$ is either prime or almost prime. The parity problem suggests that, because a prime number has an odd number of prime factors, sieve theory won't be successful in separating out the two cases.

We will here focus on one of the ways to break the parity barrier (namely the existence of Siegel zeros), even though there are now a few techniques to break that barrier. One of the most famous proofs waiving the parity problem is due to Friedlander and Iwaniec, who used a "parity-sensitive" sieve to show (see [J.F97])

Theorem 1.1.8 (Friedlander-Iwaniec,1997)**.** *There are infinitely many prime numbers of the form $a^2 + b^4$.*

But let's get back to the matter in hand : Siegel zeros. These exceptional zeros, if they exist, would prove the GRH to be false (one can now understand how strong the implications due to their existence are). We currently think that the parity problem is strongly linked to the randomness of μ (or to the randomness of its sister, λ, the Liouville function). However, if one proves that Siegel zeros exist, then one of the consequences would be that they could "waive" that randomness, and by consequence that would break the parity barrier.

It's very informal and true "in principle", for we do not fully understand that problem yet. Having said that, Heath-Brown's paper is the proof that something strange happens when one supposes the existence of these exceptional zeros, something that breaks through the parity barrier.

1.2 Siegel zeros

1.2.1 Definition and lacunarity

Let's give the definition of a Siegel zero, as defined by Heath-Brown.

Definition 1.2.1. *Let χ be a Dirichlet character (mod q) and L be the corresponding L-function. In [H.D67] is shown that there is an effectively computable numerical constant C_0 such that*

$$\forall \sigma \geq 1 - \frac{C_0}{\log(q(|t|+2))},\ L(\sigma+it,\chi) \neq 0$$

except possibly when χ is real and $t=0$. Exceptional zeros of this kind are called Siegel zeros.

What are the consequences of such zeros ? Well, first, the GRH would be false (which is kind of problematic). The universe in which Siegel zeros exist is "a parallel" one, which is very unlikely. However, in this self-consistent universe -whose constistence is still an open problem- a lot of interesting things happen. One of them is the is the fact that $L(1,\chi)$ is "small" ($L(1,\chi)\log q = o(1)$ according to [Pra19]). It also implies that the sequence $1 \star \chi$ is lacunary. Moreover, heuristically, $L(1,\chi)$ being small implies that most p must have $\chi(p) = -1$ since

$$L(1,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1}$$

This implies $\mu \approx \chi$ on the set of squarefree integers (we'll talk about that later). These are powerful pieces of information.

Concerning the lacunarity, we have

Theorem 1.2.2.

$$\sum_{n\leq x} \frac{(1\star\chi)(n)}{n} = L(1,\chi)(\log x + \gamma) + L'(1,\chi) + O\left(\frac{q\log x}{x^{1/2}}\right)$$

To prove that, we'll need a lemma concerning the summatory functions of χ/id and $\chi\log/id$.

Lemma 1.2.3. *For every $x \geq 1$ and χ non-principal character of modulus q*

$$\sum_{n\leq x}\frac{\chi(n)}{n} = L(1,\chi) + O\left(\frac{q}{x}\right),\ \sum_{n\leq x}\frac{-\chi(n)\log n}{n} = L'(1,\chi) + O\left(\frac{q\log x}{x}\right)$$

Proof (Lemma 1.2.3)**.** *We will only prove the first equality, for the second one could be proved in the same way. Let j be an integer. Since χ is non-principal*

$$\sum_{qj\leq n<(j+1)q} \chi(n) = 0$$

Thus

$$\begin{aligned}\sum_{jq\leq n<(j+1)q}\frac{\chi(n)}{n} &= \sum_{jq\leq n<(j+1)q}\chi(n)(1/n - 1/q(j+1)) \\ &\ll q\left(\frac{1}{qj} - \frac{1}{(j+1)q}\right)\end{aligned}$$

By consequence (telescoping series and rounding x and y to the nearest multiple of q), one gets, for $y > x \geq 1$

$$\sum_{x\leq n<y}\frac{\chi(n)}{n} \ll q\left(\frac{1}{y}+\frac{1}{x}\right)$$

Let's make $y\to\infty$, so that

$$L(1,\chi) = \sum_{n<x}\frac{\chi(n)}{n} + O(q/x)$$

Since, if x is an integer, $\chi(x)/x = O(q/x)$, we can change the range of summation from $n<x$ to $n\leq x$, which proves the lemma.

Proof (Theorem 1.2.2). *Since χ and 1 have "almost" the same complexity, one can study $1 \star \chi$ in the same way they would study $1 \star 1 = d$, the divisor function. Namely, we will use the Dirichlet hyperbola method.*

First remark that $(1 \star \chi)(n)/n = ((1/id) \star (\chi/id))(n)$, so that we can take $f(n) = \chi(n)/n$, $g(n) = 1/n$ and $y = \sqrt{x}$ in Theorem 3.1.5. Then

$$\begin{aligned}\sum_{n \le x} \frac{1 \star \chi(n)}{n} = &\sum_{n \le \sqrt{x}} \frac{1}{n} F(x/n) \\ &+ \sum_{n \le \sqrt{x}} \frac{\chi(n)}{n} H_{x/n} \\ &- F(\sqrt{x}) H_{\sqrt{x}}\end{aligned}$$

By Lemma 1.2.3, the first summand is

$$L(1, \chi)(\log y + \gamma) + O(q/\sqrt{x})$$

The second one is

$$(\log x + \gamma) L(1, \chi) + L'(1, \chi) + O(q \log x/\sqrt{x})$$

Finally, the third one is

$$-L(1, \chi)(\log y + \gamma) + O(q \log x/\sqrt{x})$$

From this follows an immediate consequence, that shows that the summatory function of $1 \star \chi/id$ is very small : By subtraction (and by the fact that $L(1, \chi) \gg 1/\sqrt{q}$), for A a large constant and q large enough, we have

$$\sum_{q^4 < n \le q^A} \frac{(1 \star \chi)(n)}{n} \le A \times L(1, \chi) \log q$$

But why is that concept of lacunarity important ? To answer that question, we'll take the example of Chowla's conjecture.

Conjecture (Chowla, 1965). *Let ψ be a real primitive Dirichlet character. Then*

$$L(1/2, \psi) \neq 0$$

As a lot of modern works on nonvanishing, the key point is the mollification method : for example, to study the proportion of ψ such that $L(1/2, \psi) \neq 0$, one studies the moments of the L function, using the inequality

$$\left| \sum_{\psi[q]} b_\psi \right|^2 \le \left(\sum_{\psi[q]} |b_\psi^2| \right) \left(\sum_{\psi[q], b_\psi \neq 0} 1 \right)$$

If we apply this inequality directly with $b_\psi = L(1/2, \psi)$, then we do not obtain a positive proportion result, because the first moment has size $\approx q$ while the second moment has size $\approx q \log q$, according to [Pra19].

To take advantage of lacunarity, one should actually study nonvanishing of

$$L(1/2, \psi) L(1/2, \chi\psi) = \sum_{n \ge 1} \frac{\psi(n)(1 \star \chi)(n)}{n^{1/2}}$$

The fact that a lot of the coefficient vanish is really helpful, and allows us to get better results on the density of character satisfying Chowla's conjecture.

We won't go further in that example, for it would become to far from the subject (and quite complicated), but it is enough to show what kind of consequences the existence of exceptional zeros would have.

1.2.2 The Deuring-Heilbronn phenomenon

Another fascinating consequence of Siegel zeros' existence would be a "repulsion phenomenon". By "repulsion", we mean that if a "bad" Siegel zero ($i.e$ really close to 1) exist, then it would have implications for zeros of other L-functions, and in particular, it would imply zero-free regions for these functions.

But let's be a little bit more precise. First, let's have a look again at the classical zero-free region given in Definition 1.2.1. Now, let's state how exceptional zeros would affect the zero-free region of other L-functions. Here we will give a version of the Deuring-Heilbronn phenomenon due to Linnik (but one could find many other versions of the phenomenon).

Theorem 1.2.4 (Linnik, 1944). *Suppose $q \geq 2$ is such that there is an exceptional zero $\beta = 1 - \frac{\varepsilon}{\log q}$ with ε small. Then all other zeroes $\sigma + it$ of L-functions of modulus q are such that*

$$\sigma \leq 1 - C_0 \frac{\log\left(\frac{1}{\varepsilon}\right)}{\log(q(2+|t|))}$$

The proof could be found in [Tao15]. There, one could also find a remark of T.Tao about the regularity of the spacing of zeros due to the existence of an exceptional zero. This is not the Deuring-Heilbronn phenomenon, but it has its place in this section, for it directly relies on how the spacing of zeros is impacted by the existence of a Siegel zero.

> *One particularly striking consequence of an exceptional zero $L(\beta, \chi_1)$ is that the spacing of zeroes of other L-functions become extremely regular; roughly speaking, for most other characters χ whose conductor q is somewhat (but not too much) larger than the conductor q_1 of χ_1, the zeroes of $L(s,\chi)L(s,\chi\chi_1)$ (at moderate height) mostly lie on the critical line and are spaced in approximate arithmetic progression. This phenomenon was first discovered by Montgomery and Weinberger and can roughly be explained as follows. By an approximate functional equation, one can approximately write $L(s,\chi)L(s,\chi_1)$ as the sum of $\sum_{n \lessapprox \sqrt{qq_1}} \frac{\chi(1*\chi_1)(n)}{n^s}$ plus $\sum_{n \lessapprox \sqrt{qq_1}} \frac{\bar{\chi}(1*\chi_1)(n)}{n^{1-s}}$ times a gamma factor which oscillates like $(q|t|)^{it}$ when $s = 1/2 + it$. The smallness of $1 * \chi_1(n)$ on average for medium-sized n (as suggested for instance by Bombieri's lemma) suggests that these sums should be well approximated by much shorter sums, which oscillate quite slowly in t. This gives an approximation to $L(1/2+it,\chi)L(1/2+it,\chi_1)$ that is of the form $F(t) + (q|t|)^{it}G(t)$ for slowly varying F, G, which can then be used to place the zeroes of this function in approximate arithmetic progression on the real line.*

1.3 Context for Heath-Brown's paper

1.3.1 Generalities

Now that we have the background required, it's time to give the statement of the paper (a more explicit statement can be found in Theorem 1.3.3).

Theorem 1.3.1 (Heath-Brown, 1982). *If Siegel zeros exist (in an appropriate sense), then there are infinitely many twin primes.*

The idea of splitting the proof and considering the case where Siegel zeros exist is, at first sight, quite strange. Indeed, mathematicians consider that the existence of Siegel zeros is the opposite of the Generalized Riemann Hypothesis. We are almost as sure that theses zeros don't exist as we are sure that the GRH is true. However, such an approach has already been proved to be fruitful. When Heilbronn's worked on the class number problem, he supposed that the GRH (widely believed to be true) was false to show his result. The remaining case (GRH is true) was easy to prove.

1.3.2 Skeleton of the proof

To "measure" how often a pair of linear forms will be simultaneously prime, Heath-Brown uses the Λ function and proves

Theorem 1.3.2. *We write* $l_i(n) := \alpha_i n + \beta_i$, $(i = 1, 2)$. *We suppose these forms satisfy a set of conditions we'll formally define later. Let's keep the previous notations about* q, χ, β_0 *and suppose that* $\eta := ((1 - \beta_0)\log q)^{-1} \geq 3$. *Then, if* $\alpha = (\alpha_1, \alpha_2)$, *we have*

$$\sum_{x<n\leq 2x} \Lambda(l_1(n))\Lambda(l_2(n)) = \mathfrak{S}C(\alpha)x + O(x\log_2^{-1}\eta)$$

uniformly for $q^{250} \leq x \leq q^{500}$, *where*

$$\mathfrak{S} := 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)$$

and

$$C(\alpha) := 2\prod_{p|\alpha, p>2}\left(1 - \frac{2}{p}\right)^{-1}$$

The implied constant depends on α_i, β_i *only.*

Another theorem, easily deduced from the one above, is :

Theorem 1.3.3. *There exist a constant* C_1 *such that at least one of the following assertions holds :*

1. *Every pair* (l_1, l_2) *as defined earlier represents primes simultaneously for infinitely many integers*
2. *For all* $q \geq 2$ *and all* χ *(mod q), we have* $L(\sigma + it, \chi) \neq 0$ *for*

$$\sigma \geq 1 - \frac{C_1}{\log(q|t| + 2)}$$

These results show that Λ is a good function to "measure" primality (and the PNT already shown that). But as the parity problem suggests, we won't be able to get a sharp result if we do not get rid of the randomness of μ.

The idea of Heath-Brown, to tackle the randomness caused by μ, is to replace the Van Mangold function Λ by $\tilde{\Lambda} := (\mu^2\chi) \star \log$. This idea is justified by the following lemma, that shows that if a Siegel zero exists and if η is large, then a lot of primes are such that $\chi(p) = -1$:

Lemma 1.3.4. *If we use the notations already introduced, then :*

$$\sum_{p\leq x, \chi(p)=1} p^{-1}\log p \ll \log q(\log\eta)^{-1/2}$$

for $q^{250} \leq x \leq q^{500}$.

This shows the fundamental underlying principle -that justifies the modification of Van Mangold function- of this proof : $\mu^2\chi \approx \mu$.

Now, the aim is to formally show that

$$\sum_{x<n\leq 2x} \Lambda(l_1(n))\Lambda(l_2(n)) \approx \sum_{x<n\leq 2x} \tilde{\Lambda}(l_1(n))\tilde{\Lambda}(l_2(n))$$

and then, we'll have to estimate the right term.

2 Heath-Brown's paper

2.1 Notations

For this section, let β_0 be a Siegel zero of $s \mapsto L(s,\chi)$, where χ is a real character of conductor $q \geq 2$. We fix $q^{250} \leq x \leq q^{500}$ and $2 \leq z \leq q$. As usual, μ is the Möbius function, Λ is the Van Mangold function and ϕ is Euler's totient function. p will always be a prime, and n an integer. By $p^e || n$, we shall mean $p^e | n$ and $p^{e+1} \not| n$. The letter A will denote an effective positive constant depending on α_i, β_i (Be careful, it might not be the same at each occurrence). Finally, we write $L := \log q$ and we keep the notation $\eta = ((1-\beta_0)L)^{-1}$.

Let's define the set of conditions on l_i we talked about earlier.

- → $\alpha_i \in \mathbb{N},\ \beta_i \in \mathbb{Z}$
- → $(\alpha_i, \beta_i) = 1$
- → $2 | \alpha_i$
- → $\alpha_1\beta_2 - \beta_1\alpha_2 \neq 0$
- → $p | (\alpha_1\beta_2 - \beta_1\alpha_2) \Rightarrow p | \alpha_i$
- → $p | \alpha_1 \Leftrightarrow p | \alpha_2$
- → $p | \alpha_i \Rightarrow p^2 | \alpha_i$

From now and until the end of the section, we'll write l_i instead of $l_i(n)$, l instead of $l_1 l_2$, we suppose that $\eta \geq 3$ and we suppose that l_i satisfy the set of conditions defined above.

We also define

$$P := \prod_{\substack{2<p<z \\ \chi(p)=1}} p$$

and three Λ-sums [1]

$$S^{(0)} := \sum_{x<n\leq 2x} \Lambda(l_1)\Lambda(l_2),\ S^{(1)} := \sum_{\substack{x<n\leq 2x \\ (l,qP)=1}} \Lambda(l_1)\Lambda(l_2),\ S^{(2)} := \sum_{\substack{x<n\leq 2x \\ (l,qP)=1}} \tilde{\Lambda}(l_1)\tilde{\Lambda}(l_2)$$

We finally introduce another Λ-related function and a fourth Λ-sum

$$\Lambda^*(n) := \sum_{d\in\mathscr{D}(n)} \chi(d)\log(n/d)$$

with $\mathscr{D}(n)$ the set $\{d|n,\ [p<z \textit{ and } \chi(p)=-1] \Rightarrow p^2 \nmid d\}$. The related Λ-sum is

$$S^{(3)} := \sum_{\substack{x<n\leq 2x \\ (l,qP)=1}} \Lambda^*(l_1)\Lambda^*(l_2)$$

2.2 First lemmas

Since the paper is very dense and full of references, we won't detail every single proof. The ambition of this section is more about giving an idea of the proof. If one wants to have a complete proof of the statements we'll make, every detail can be found in [DB82].

2.2.1 Estimation of Λ-sums

A first, and not too hard to obtain, estimate is the one that links $S^{(0)}$ and $S^{(1)}$: $S^{(0)}=S^{(1)} + O(L^4 z)$. Further analysis would lead us to a result linking $S^{(1)}$ and $S^{(2)}$ this time

Lemma 2.2.1. *Define* $z_0 := L/\log z$. *If* $z_0 \leq L^{1/3}$, *then we have*

$$S^{(1)} = S^{(2)} + 0(xz_0^{-1}) + O\left(x^{-1}\exp(Az_0)\sum_{p\leq x, \chi(p)=1} p^{-1}\log p\right)$$

---

[1] we recall that $\tilde{\Lambda} := (u^2\chi) \star \log$

The proof will need two more lemmas. The first one states that

Lemma 2.2.2. *If* $(n,q)=1$*, then* $\Lambda^*(n)\geq 0$ *and*

$$0\leq\tilde{\Lambda}(n)-\Lambda(n)\ll f(n)\log(n)+(f(n_+)-1)\Lambda(n_-)$$

where

$$n_+=\prod_{p^e||n,\chi(p)=1}p^e,\quad n_-=\prod_{p^e||n,\chi(p)=-1}p^e,\quad f=\mu^2\chi\star 1$$

The other one is an application of a Selberg sieve.

Lemma 2.2.3. *Let* $Z\ll\left(\frac{x}{d_1d_2}\right)^2$*. Then*

$$S(d_1,d_2;Z):=\#\left\{x<n\leq 2x: d_i|l_i,\ \left(l_i/d_i,\prod_{p\leq z}p\right)=1\right\}\ll\frac{x}{\phi(d_1d_2)\log^2 Z}$$

Proof (Lemma 2.2.3). *We'll use a slightly different result than the one stated in the first section (but the idea is the same). The modification can be found in [H.H74]. We assume that* $(d_i,\alpha_i)=1$ *and* $(d_1,d_2)=1$ *(else,* $S(d_1,d_2;Z)=0$*). Let* $\mathscr{P}$ *be the set of all primes and*

$$\mathscr{A}:=\{l/d_1d_2: x<n\leq 2x, d_i|l_i\}$$

Then, if k *is square-free, we have*

$$\#\{a\in\mathscr{A}:k|a\}=\sum_{k=k_1k_2}\#\{n:x<n\leq 2x, d_i|l_i\}$$

The summands on the right are zero unless $(k_1d_1,k_2d_2)=1$ *and* $(k_i,\alpha_i)=1$*. In the latter case, there is a unique congruence class for* $n(mod\ k_1d_1k_2d_2)$ *such that* $k_id_i|l_i$*. Thus*

$$\#\{a\in\mathscr{A}:k|a\}=N(k)\left(\frac{x}{d_1d_2}+O(1)\right)$$

where N *is a multiplicative function such that* $N(p)=2$ *if* $p\nmid d_1d_2\alpha$*,* $N(p)=1$ *if* $p|d_1d_2$ *and* $N(p)=0$ *if* $p|\alpha$*. Then, by [H.H74], we have*

$$\begin{aligned}S(d_1,d_2;Z)&\leq S(d_1,d_2;Z^{1/5})\\&\ll\frac{x}{d_1d_2}W(Z^{1/5})+\sum_{d<Z^{2/5}}\mu^2(d)3^{\omega(d)}N(d)\end{aligned}$$

where $W(X)=\prod_{p\leq X}(1-N(p)/p)$*. But we have that*

$$\begin{aligned}W(Z^{1/5})&\ll\prod_{3\leq p\leq Z^{1/5}}(1-2/p)\prod_{d|d_1d_2}\frac{1-1/p}{1-2/p}\\&\ll(\log Z)^{-2}d_1d_2\phi(d_1d_2)^{-1}\end{aligned}$$

and

$$\begin{aligned}\sum_{d<Z^{2/5}}\mu^2(d)3^{\omega(d)}N(d)&\ll\sum_{p<Z^{2/5}}d^{\varepsilon}\\&\ll Z^{1/2}(\log Z)^{-2}\\&\ll(x/d_1d_2)(\log Z)^{-2}\end{aligned}$$

These together prove the lemma.

Let's now prove Lemma 2.2.1, for the proof shows how one could use sieves to get really good estimates.

Proof (Lemma 2.2.1). *We first suppose that either l_1 or l_2 has a factor $p^e \geq z$, where $e \geq 2$ (say l_1). If $(n, 2qP) = 1$ then*

$$f(n) \leq f(n_+) \ll \exp(Az_0) \tag{1}$$

since $p|n_+ \Rightarrow p \geq z$. Hence, Lemma 2.2.2 yields

$$\tilde{\Lambda}(l_1)\tilde{\Lambda}(l_2) \ll L^2 \exp(Az_0)$$

Thus, for each $e \geq 2$

$$\begin{aligned}\sum_{p^e \geq z} \sum_{\substack{x<2n\leq 2x \\ p^e|l_1,(l,qP)=1}} \tilde{\Lambda}(l_1)\tilde{\Lambda}(l_2) &\ll L^2 \exp(Az_0)xz^{-1/2} \\ &\ll xz_0^{-1}L^{-1}\end{aligned}$$

since $z_0 \leq L^{1/3}$. A same bound holds if the linear form with that kind of factor is l_2. We now can write

$$S^{(2)} - S^{(1)} \ll xz_0^{-1} + \sum \tilde{\Lambda}(l_1)\tilde{\Lambda}(l_2) - \Lambda(l_1)\Lambda(l_2)$$

where the sum here is subject to the conditions : $\tilde{\Lambda}(l_i) \neq 0$, $(l, qP) = 1$, $x < n \leq 2x$ and l_i has no factor $p^e \geq z$, with $e \geq 2$.

Define

$$R := \prod_{p<z} p$$

and

$$U := \{n \geq z : (n, qR) = 1\}, \quad V := \{p^e \leq z : \chi(p) = -1\}, \quad W := V \cup \{1\}$$

If n satisfies the conditions above, then Lemma 2.2.2 shows that l_i must be in one of the sets

$$L_1 := \{p : p \nmid qR\}$$

$$L_2 := \{pv : \chi(p) = 1, p \geq z, v \in V\}$$

$$L_3 := \{puw, \chi(p) = 1, p \geq z, u \in U, w \in W\}$$

Write

$$L_4 := \{uw : u \in U, w \in W\}$$

and then, if $\tilde{\Lambda}(l_1)\tilde{\Lambda}(l_2) \neq \Lambda(l_1)\Lambda(l_2)$, either (l_1, l_2) or (l_2, l_1) will be in one of the sets $L_2 \times (L_1 \cup L_2)$ or $L_3 \times L_4$.

Let's only treat the case where $(l_1, l_2) = (pu_1w_1, u_2w_2) \in L_3 \times L_4$, and we'll admit that the contribution of all the other cases is

$$\ll xz_0^{-1} + xL^{-1}\exp(Az_0) \sum_{p\leq x,\chi(p)=1} \log p/p$$

Define $L(w) = L$ if $w = 1$ and $L(w) = \Lambda(w)$ otherwise, so that, by (1)

$$\tilde{\Lambda}(l_1)\tilde{\Lambda}(l_2) - \Lambda(l_1)\Lambda(l_2) \ll L(w_1)L(w_2)\exp(Az_0)$$

Since $u_1 \geq z$, we have $p \leq x/zw_1$. Now if $w_2 = p^e \geq z^{1/2}$, $e \geq 2$, then, as shown at the beginning of the proof, the contribution to $S^{(2)} - S^{(1)}$ is

$$\ll L^3 \exp(Az_0)xz^{-1/4} \ll xz_0^{-1}$$

If $w_2 = p \geq z^{1/2}$, then the contribution made is

$$\ll \sum_{\substack{z\leq p\leq x \\ \chi(p)=1}} \sum_{\substack{w\in W \\ w\ll x/zp}} L(w)L\exp(Az_0)S(pw, 1; z^{1/2}) \tag{2}$$

The remaining case yields

$$\ll \sum_{\substack{z\le p\le x\\ \chi(p)=1}} \sum_{\substack{w_i\in W\\ w_1\ll x/zp,\ w_2\le z^{1/2}}} L(w_1)L(w_2)\exp(Az_0)S(pw_1,w_2;z) \tag{3}$$

However, in (2), $z^{1/2} \ll (x/pw)^2$ *and in (3)* $z \ll (x/pw_1w_2)^2$*. Thus, thanks to Lemma 2.2.3, it is now easy to conclude.*

Finally, Lemma 1.3.4 together with Lemma 2.2.1 implies a way simpler estimation (if $z_0 \le A\log\log\eta$)

$$S^{(1)} = S^{(2)} + O(xz_0^{-1})$$

It's now time to prove the estimation we wanted from the beginning

Proposition 2.2.4. *If* $z_0 \le A\log\log\eta$*, then* $S^{(0)} = S^{(3)} + O(xz_0^{-1})$*.*

Proof (Proposition 2.2.4)**.** *If* $(l_i, qP) = 1$*, then* $\Lambda^*(l_i) = \tilde{\Lambda}(l_i)$ *except when* $p^2|l_i$ *for some* $p \ge z$*. Since* $\Lambda^*(l_i), \tilde{\Lambda}(l_i) \ll L2^{\omega(l_i)}$*, we have*

$$\begin{aligned} S^{(2)} - S^{(3)} &\ll \sum_{p\ge z} \sum_{\substack{x<n\le 2x\\ p^2|l_i}} L^2 2^{\omega(l_1)+\omega(l_2)} \\ &\ll xL^2\exp(AL\log^{-1}L)z^{-1} \\ &\ll xz_0^{-1} \end{aligned}$$

if $z_0 \le A\log\log\eta$*. Since* $z_0 \le A\log\log\eta$ *implies* $z_0 \le L^{1/3}$*, we can use Lemma 2.2.1 and together, the estimations given previously lead to the wanted equality.*

2.2.2 L-functions

In this section, we'll talk about two more results, proved thanks to the theory of L-functions. The first one is Lemma 1.3.4, and since it's the main idea behind Heath-Brown's theorem, we'll prove it.

Proof (Lemma 1.3.4)**.** *We first use a formula from [H.D67]. We use* $s = 1+L^{-1}$ *and* $s' = 1+aL^{-1}$*, where* $a \gg 1$*. We have*

$$\frac{L'(s',\chi)}{L(s',\chi)} - \frac{L'(s,\chi)}{L(s,\chi)} = \sum\left(\frac{1}{s'-\rho} - \frac{1}{s-\rho}\right) + O(1)$$

For $\rho = \beta_0$*, we have that* $(s'-\rho)^{-1} \ll La^{-1}$*, and we always have* $s'-s \ll aL^{-1}$*. The sum above is then*

$$\begin{aligned} &= -\frac{1}{s-\beta_0} + O(La^{-1}) + O\left(\sum_{\rho\ne\beta_0} \frac{|s'-s|}{|(s'-\rho)(s-\rho)|}\right) \\ &= -\frac{1}{s-\beta_0} + O(La^{-1}) + O(aL^{-1}\sum_{\rho\ne\beta_0}|\rho-1|^{-2}) \end{aligned}$$

By [H.D67], the contribution from zeros $\rho = \beta + i\gamma$ *with* $\gamma \ge 1$ *is*

$$\ll aL^{-1}\sum_{|\gamma|\ge 1}|2-\rho|^{-2} \ll a$$

A result of Prachar ([K.P57]) tells us that the number of zeros in a disc $|s-1| \le r$ *is* $\ll 1 + r\log q$ *when* $r \le 2$*. Consequently, the contribution from zeros* $\rho \ne \beta_0$*,* $|\gamma| \le 1$ *is*

$$\ll aL^{-1}(r_0^{-2} + Lr_0^{-1})$$

where

$$r_0 = \min\{|1-\rho| : \rho \neq \beta_0\}$$

By the Deuring-Heilbronn phenomenon (see [M.J77]), one has that $r_0 \gg L^{-1}\log\eta$*. Thus*

$$aL^{-1}\sum_{\substack{\rho\neq\beta_0\\|\gamma|\leq 1}}|\rho-1|^{-2} \ll aL(\log\eta)^{-1}$$

Finally, we have

$$\frac{L'(s',\chi)}{L(s',\chi)} \ll \sum_{1}^{\infty}\Lambda(n)n^{-s'} \ll (s'-1)^{-1} \ll La^{-1}$$

whence, on comparing estimates, we have

$$-\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{s-\beta_0} + O(La^{-1}) + O(aL(\log\eta)^{-1})$$

We make the optimal choice $a = (\log\eta)^{1/2}$*. Again by [H.D67], one has*

$$\frac{\zeta'(2)}{\zeta(2)} - \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + \sum\left(\frac{1}{2-\rho} - \frac{1}{s-\rho}\right) + O(1)$$

for the same s *as above (but* ρ *runs over zeros of* ζ*). Thus*

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + O(1)$$

since $\sum|\rho|^{-2}$ *converges. Since*

$$\sum_{\substack{p\leq x\\\chi(p)=1}}\frac{\log p}{p} \ll \sum_{\substack{n\leq x\\\chi(n)=1}}\frac{\Lambda(n)}{n} \ll \sum_{\substack{n=1\\\chi(n)=1}}^{\infty}\frac{\Lambda(n)}{n^s} \ll -\frac{\zeta'(s)}{\zeta(s)} - \frac{L'(s,\chi)}{L(s,\chi)}$$

and

$$\frac{1}{s-1} - \frac{1}{s-\beta_0} = \frac{1-\beta_0}{(s-1)(s-\beta_0)} \ll \frac{(\eta L)^{-1}}{(s-1)^2} \ll L\eta^{-1} \ll L(\log\eta)^{1/2}$$

the proof is now complete.

The other lemma whose proof relates to L-functions states that

Lemma 2.2.5.

$$\frac{L'(1,\chi)}{L(1,\chi)} = \eta L + O(L(\log\eta)^{-1/2})$$

whenever $z_0 \leq A\log\log\eta$*.*

2.3 Rosser's sieve

As one could find in [H.I80], for $d \in \mathbb{N}$ and $D \geq 2$, there exist weights $\lambda_d^{\pm}(D)$, depending only on d, D and the ± sign, with the properties $\lambda_1^{\pm}(D) = 1$, $|\lambda_d^{\pm}(D)| \leq 1$ for all d,

$$\pm\sum_{d|n}\lambda_d^{\pm}(D) \geq 0 \ (n > 1)$$

and $\lambda_d^{\pm}(D) = 0$ for $d \geq D$. Since $\lambda^*(l_1)\lambda^*(l_2) \geq 0$, we have

$$\sum_{d|P}\lambda_d^{-}(D)S(d) \leq S^{(3)} \leq \sum_{d|P}\lambda_d^{+}(D)S(d) \tag{4}$$

where $D \geq 2$ and

$$S(d) := \sum_{\substack{x<n\leq 2x\\d|l,(l,q)=1}}\lambda^*(l_1)\lambda^*(l_2)$$

The goal of the section is now to prove

Proposition 2.3.1. *Let $d|P$, $(d,\alpha)=1$ and $d,z\le q^{1/3}$. Then*

$$S(d)=\kappa\frac{G(d)}{d}\left\{\left(\frac{L'(1,\chi)}{L(1,\chi)}\right)^2+A^2(d)+A'(d)+C_2\right\}+O(xL^4z^{-1}d^{-1}4^{\omega(d)})$$

Here

$$\kappa=xL(1,\chi)^2\prod_{p|q,p\nmid\alpha}\left(1-\frac{2}{p}\right)\prod_{p|\alpha}\left(1-\frac{\chi(p)}{p}\right)^2\prod_{\substack{p\nmid\alpha\\ \chi(p)=1}}\left(1-\frac{1}{p^2}\right)\prod_{\substack{p\nmid\alpha\\ \chi(p)=-1}}\left(1-\frac{2}{p}\right)\left(1+\frac{1}{p}\right)^2$$

and

$$G(d)=2^{\omega(d)}\prod_{p|d}\left(\frac{2p-1}{p+1}\right)$$

Moreover, A,A' are additive functions for which

$$A(p)\ll\log p,\ \ A'(p)\ll B\log p\ \ \left(B:=L+\left|\frac{L'(1,\chi)}{L(1,\chi)}\right|\right)$$

and C_2 is independent of d and satisfies

$$C_2\ll BL$$

The proof of that equality is the most difficult and the longest one of [DB82], and we'll have to cut it into small pieces so that it is as readable as possible.

2.3.1 Preliminary steps

Let's define

$$S(\delta_1,\delta_2;V_1,V_2):=\sum_{\substack{x<n\le 2x\\ (l,q)=1,\delta_i|l_i}}\left(\sum_{\substack{w_1v_1=l_1/\delta_1\\ v_1>V_1}}\chi(w_1)\right)\left(\sum_{\substack{w_2v_2=l_2/\delta_2\\ v_2>V_2}}\chi(w_2)\right)$$

By re-writing λ^* (thus re-writing $S(d)$), one shows

Lemma 2.3.2. *Let $d|P$ and $Q:=\prod_{p<z,\chi(p)=-1}p$. Then*

$$\begin{aligned}S(d)=&\sum_{\substack{m_i|Q\\ m_i<q}}\mu(m_1)\mu(m_2)\sum_{d=d_1d_2}\sum_{j_ik_i|d_i}\mu(j_1)\mu(j_2)\\ &\times\int_{(j_1k_1)^{-1}}^{\infty}\int_{(j_2k_2)^{-1}}^{\infty}S(m_1^2d_1j_1,m_2^2d_2j_2;V_1,V_2)\frac{dV_2}{V_2}\frac{dV_1}{V_1}+O(x^{1+\varepsilon}q^{-1})\end{aligned}$$

The idea of Heath-Brown is then to write $S(\delta_1,\delta_2;V_1,V_2)$ according to the residue class (mod q) of v_i,w_i. Moreover, it will be useful to break the ranges of summation into intervals $R_i<v_i\le 2R_i, S_i<w_i\le 2S_i$ where R_i/V_i and S_i are powers of 2. Since $l_i\asymp x$, we may assume

$$R_i\ge V_i,\ \ S_i\gg 1,\ \ \delta_iS_iR_i\asymp x \tag{5}$$

Thus

$$S(\delta_1,\delta_2;V_1,V_2)=\sum_{R_i,S_i}\sum_{a_i,b_i=1}^{q}\chi(b_1b_2)S$$

where $(q,a_ib_i)=1$ and

$$S=\#\{(v_i,w_i):R_i<v_i\le 2R_i,S_i<w_i\le 2S_i,\delta_iv_iw_i=l_i,x<n\le 2x,v_i\equiv a_i,w_i\equiv b_i[q]\}$$

By re-writing the conditions for S, to remove the variables n and v_1, one can show

$$S = \sum_{\substack{S_i < w_i \le 2S_i \\ w_i \equiv b_i[q]}} \#\{T_1 < v_2 \le T_2 : v_2 w_2 \equiv C[D\delta_1 w_1]\}$$

where $D = \alpha_2 q \Delta^{-1}$, with $\Delta := (\alpha_1, q)$, and C is a constant that does not depend on v_2, w_2 and satisfies $(C, D\delta_1 w_1) = 1$. Moreover

$$T_1 = \max\left(R_2, \frac{\alpha_2 x + \beta_2}{\delta_2 w_2}, \frac{\alpha_2\delta_1 w_1 R_1 + \alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1\delta_2 w_2}\right)$$

$$T_2 = \min\left(2R_2, \frac{2\alpha_2 x + \beta_2}{\delta_2 w_2}, \frac{2\alpha_2\delta_1 w_1 R_1 + \alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1\delta_2 w_2}\right)$$

In the next section, we'll prove that

Lemma 2.3.3. *Let $(C,k) = 1, q|k$ and $(q,b) = 1$. Define $\bar{n}$ by $0 < \bar{n} \le k$, $n\bar{n} \equiv 1[k]$. Let I be a subinterval of $(E, 2E]$ where $E \ge 1$, and let $T \in \mathbb{R}$. Then*

$$\sum_{\substack{n \in I \\ (n,k)=1 \\ n \equiv b[q]}} \psi(f(n)) \ll (1 + |T|E^{-1}k^{-1})(E + k)q^{3/2}k^{\varepsilon - 1/4}$$

for $f(n) = (T - C\bar{n})/k$ or $f(n) = (T/n - C\bar{n})/k$.

But let's admit it for now. If $(w_2, D\delta_1 w_1) = 1$, we write $\bar{w_2}$ for the integer such that $w_2\bar{w_2} \equiv 1[D\delta_1 w_1]$ and $0 < \bar{w_2} \le D\delta_1 w_1$. Then, we have

$$\#\{T_1 < v_2 \le T_2 : v_2 w_2 \equiv C[D\delta_1 w_1]\} = (T_2 - T_1)(D\delta_1 w_1)^{-1} + \psi\left(\frac{T_1 - C\bar{w_2}}{D\delta_1 w_1}\right) + \psi\left(\frac{T_2 - C\bar{w_2}}{D\delta_1 w_1}\right)$$

We take $k = D\delta_1 w_1$ and $E = S_2$, and we break the range of summation for w_2 into $O(1)$ parts so that $T_i = T$ or T/w_2 on each part. Since

$$T \ll R_2 + x/\delta_2 + \delta_1 w_1 R_1/\delta_2 \ll x/\delta_2$$

Lemma 2.3.3 yields

$$\sum_{w_2} \psi\left(\frac{T_i - C\bar{w_2}}{D\delta_1 w_1}\right) \ll (1 + x/(\delta_2\delta_1 D S_2 w_1))(S_2 + D\delta_1 w_1)q^{3/2}(D\delta_1 w_1)^{\varepsilon - 1/4}$$

and the total contribution to S is

$$\begin{aligned} &\ll S_1(1 + x/(\delta_1\delta_2 S_1 S_2))(S_2 + S_1)\delta_1 q^{5/2}(\delta_1 S_1)^{\varepsilon - 1/4} \\ &\ll \delta_1 q^{5/2}(S_1 S_2 + S_1^2 + x + xS_1/S_2)S_1^{\varepsilon - 1/4} \end{aligned}$$

Suppose that $S_i \le R_i$, $(i = 1, 2)$. Since $\delta_i S_i R_i \asymp x$, we have $S_i \ll \sqrt{x}$. Thus the above is

$$\ll \delta_1 q^{5/2} x^{15/16 + \varepsilon}$$

if $S_1 S_2 \gg x^{15/16}$. Since there are $O(S_1 S_2)$ terms in the sum, and since ψ contributes to $O(1)$ for each one, it stays true if $S_1 S_2 \ll x^{15/16}$. All in all

Lemma 2.3.4. *We have*

$$S(\delta_1, \delta_2; V_1, V_2) = \sum_{R_i, S_i} \sum_{a_i, b_i = 1}^{q} \chi(b_1 b_2) S \tag{6}$$

where R_i/V_i and S_i runs over powers of 2 subject to (5), and a_i, b_i are such that

$$(q, a_i b_i) = 1,\ \delta_i a_i b_i \equiv \beta_i[\Delta],\ \alpha_1(\delta_2 a_2 b_2 - \beta_2) \equiv \alpha_2(\delta_1 a_1 b_1 - \beta_1)[q\alpha]$$

Moreover, if we suppose that $S_i \leq R_i, \ (i = 1,2)$*, then*

$$S = \sum_{\substack{S_i < w_i \leq 2S_i \\ w_i \equiv b_i [q]}} (T_2 - T_1)(D\delta_1 w_1)^{-1} + O(\delta_1 q^{5/2} x^{15/16+\varepsilon}) \tag{7}$$

where the summation indicates that $T_2 > T_1$*, that* $(w_2, D\delta_1 w_1) = 1$ *and that*

$$(w_1, \alpha) = (w_1, \delta_2) = (w_1, q) = 1$$

2.3.2 Kloosterman sums and Lemma 2.3.3

This section concerns the idea of reducing an estimation to a question involving exponential sums. This "trick" being quite common in number theory, we will take the time to prove Lemma 2.3.3

Proof (Lemma 2.3.3)**.** *We will have to use Estermann's bound ([T.E61]) for the Kloosterman sum, namely*

$$S(k; u, v) = \sum_{\substack{n=1 \\ (n,k)=1}}^{k} e\left(\frac{un + v\bar{n}}{k}\right) \ll d(k) k^{1/2} (k, u, v)^{1/2} \tag{8}$$

where $d = 1 \star 1$ *is the divisor function and* $e(x) = \exp(2\pi i x)$*.*

We consider, for $m > 0$

$$S_m := \sum_{\substack{n \in I \\ (n,k)=1 \\ n \equiv b[q]}} e(mf(n))$$

In the case where $f(n) = (T/n - C\bar{n})/k$ *we find, by partial summation, that*

$$S_m \ll (1 + m|T|E^{-1}k^{-1}) \left| \sum_{\substack{n \in I_0 \\ (n,k)=1 \\ n \equiv b[q]}} e(Cm\bar{n}/k) \right| \tag{9}$$

where $I_0 \subset I$*. This is also true in the case* f *is of the other form.*

We now write $g(n) = e(Cm\bar{n}/k)$*. Then*

$$\begin{aligned}
\sum_{\substack{n \in I_0 \\ (n,k)=1 \\ n \equiv b[q]}} g(n) &= \sum_{\substack{r=1 \\ (r,k)=1}}^{k} g(r) \sum_{\substack{n \in I_0 \\ (n,k)=1 \\ n \equiv b[q] \\ k|(r-n)}} 1 \\
&= \sum_{\substack{r=1 \\ (r,k)=1}}^{k} g(r) \sum_{\substack{n \in I_0 \\ q|(n-b)}} \frac{1}{k} \sum_{s=1}^{k} e(s(r-n)/k) \\
&= \frac{1}{k} \sum_{s=1}^{k} S(k; s, Cm) \sum_{\substack{n \in I_0 \\ q|(n-b)}} e(-sn/k)
\end{aligned}$$

Writing $k_0 := k/q$ *and* $||\theta||$ *for the distance from* θ *to* $\mathbb{Z}$*, we have, by (8)*

$$\begin{aligned}\sum_{\substack{n\in I_0\\(n,k)=1\\n\equiv b[q]}} g(n) &\ll k^{-1}d(k)(kq)^{1/2}q\sum_{s=1}^{k_0}(k_0,s,Cm)^{1/2}\min(E,||s/k_0||^{-1})\\ &\ll d(k)k^{-1/2}q^{3/2}\left(E(k_0,Cm)^{1/2}+\sum_{s=1}^{k_0-1}(k_0,s)^{1/2}||s/k_0||^{-1}\right)\\ &\ll d(k)k^{-1/2}q^{3/2}\left(E(k_0,Cm)^{1/2}+\sum_{s=1}^{k_0}k_0(k_0,s)^{1/2}s^{-1}\right)\end{aligned}$$

However

$$\begin{aligned}\sum_{s=1}^{k_0}(k_0,s)^{1/2}s^{-1} &\ll \sum_{d|k_0}d^{1/2}\sum_{\substack{1\le s\le k_0\\d|s}}s^{-1}\\ &\ll \sum_{d|k_0}d^{1/2}d^{-1}\log(2k_0)\\ &\ll d(k_0)\log(2k_0)\end{aligned}$$

By consequence, (9) yields

$$S_m \ll (1+m|T|E^{-1}k^{-1})d(k)^2q^{3/2}\log(2k_0)(E(k_0,m)^{1/2}+k_0)k^{-1/2}$$

since $(C,k_0)=1$*. Similarly, one can get*

$$\sum_{M<m\le 2M}(k_0,m)^{1/2}\ll Md(k_0)$$

Thus

$$\sum_{M<m\le 2M}|S_m| \ll (1+M|T|E^{-1}k^{-1})d(k)^3\log(2k_0)q^{3/2}M(E+k)k^{-1/2} \tag{10}$$

As stated by Heath-Brown, one can also remark that

$$\psi(\theta) = -\sum_{0<|m|\le K}\frac{e(m\theta)}{2\pi im}+O\left(\min\left(\frac{1}{K||\theta||},1\right)\right)$$

and that for $K\ge 2$ *we have*

$$\min\left(\frac{1}{K||\theta||},1\right)=\sum_{-\infty}^{\infty}a_me(m\theta)$$

$$a_m\ll\min\left(\frac{\log K}{K},\frac{K}{m^2}\right)$$

From these, we have

$$\sum_{\substack{n\in I\\(n,k)=1\\n\equiv b[q]}}\psi(f(n))\ll(\log K)\left(EK^{-1}+\sum_{1\le m\le K}m^{-1}|S_m|+\sum_{m>K}Km^{-2}|S_m|\right)$$

If $m\ge Kk^{1/2}$*, we apply that* $S_m\ll E$*. Otherwise we apply (10). All in all*

$$\sum_{\substack{n\in I\\(n,k)=1\\n\equiv b[q]}}\psi(f(n))\ll d^3(k)(\log Kk)^3(EK^{-1}+(1+K|T|E^{-1}k^{-1})q^{3/2}(E+k)k^{-1/2})$$

We choose $K=2+k^{1/4}$ *and the claim follows.*

2.3.3 Leading terms

Let's get back to where we stopped with Lemma 2.3.4. We stay in the case where the conditions of that lemma are true. The summand in (7) is

$$\underbrace{(\delta_1\delta_2 q\Delta^{-1})}_{K:=}{}^{-1}(w_1w_2)^{-1}\underbrace{\mathrm{mes}\{t\in\mathbb{R}, x\le t\le 2x, R_i\le l_i(t)/(\delta_i w_i)\le 2R_i\}}_{A(w_1,w_2):=}$$

By counting the solutions of a system of congruence equations, one shows that the contributions of the summands to $S(\delta_1,\delta_2;V_1,V_2)$ is

$$\underbrace{\prod_{\substack{p|q\\ p\nmid\alpha}}\left(1-\frac{2}{p}\right)}_{M:=}(\delta_1\delta_2)^{-1}\underbrace{\sum_{w_i}\chi(w_1w_2)(w_1w_2)^{-1}A(w_1,w_2)}_{F(R_i,S_i):=}$$

Now, let's consider the terms of (6) for which $S_1\le R_1$ and $S_2>R_2$. By writing the leading contribution to S as

$$\sum_{w_1,v_2}K^{-1}(w_1v_2)^{-1}A'$$

where A' has a definition close to that of A. We'll now explicitly solve the system of congruences, just to show at least once how we can get to the bound we want. When we sum over a_i,b_i, we necessarily have $a_2,b_1\equiv v_2,w_1[q]$. Thus we need to compute $\sum_{a_1,b_2}\chi(b_2)$ where

$$(a_1,q)=1,\ \delta_1w_1a_1\equiv\beta_1[\Delta],\ \delta_2v_2b_2\equiv\beta_2[\Delta],\ \alpha_1(\delta_2v_2b_2-\beta_2)\equiv\alpha_2(\delta_1w_1a_1-\beta_1)[q\alpha]$$

One can show that each b_2 determines a single value of a_1 (satisfying $a_1\equiv c'b_2+b'[q/\Delta]$, for certain c',b'). Then

$$\sum_{a_1,b_2}\chi(b_2)=\sum_{\substack{\delta_2v_2b_2\equiv\beta_2[\Delta]\\(b_2+b'',q/\Delta)=1}}\chi(b_2)\tag{11}$$

Let b_0 be a solution of the first condition of summation in the right term. Then the previous equality is

$$\sum_{\substack{b_2=1\\ \Delta|(b_2-b_0)}}\sum_{\substack{d|q/\Delta\\ d|(b_2+b'')}}\chi(b_2)\mu(d)=\sum_{d|q/\Delta}\mu(d)\sum\chi(b_2)$$

where the final sum is subject to b_2 running over a congruence class (mod $d\Delta$). Since χ is primitive, that sum vanishes unless $d\Delta=q$. Thus, (11) is $O(1)$. By Lemma 2.3.4, the contribution to $S(\delta_1,\delta_2;V_1,V_2)$ arising from terms with $S_1\le R_1, S_2>R_2$ is

$$\begin{aligned}&\ll\sum_{R_i,S_i}\sum_{w_1,v_2}K^{-1}(w_1v_2)^{-1}A'\\&\ll\sum_{R_i,S_i}(\delta_1\delta_2 q)^{-1}x\\&\ll xL^4(q\delta_1\delta_2)^{-1}\end{aligned}$$

Same arguments apply when $S_2\le R_2, S_1>R_1$. Finally, even if the method is slightly different (but globally the same), we get the same bound for the contribution of the term arising with both $S_i>R_i$.

All in all

$$S(\delta_1,\delta_2;V_1,V_2)=\sum_{S_i\le R_i}M(\delta_1\delta_2)^{-1}F(R_i,S_i)+O(xL^4(q\delta_1\delta_2)^{-1})+O(\delta_1\delta_2q^{13/2}x^{15/16+\varepsilon})\tag{12}$$

Since $\sum_{S_i>R_i}M(\delta_1\delta_2)^{-1}F(R_i,S_i)\ll\delta_1\delta_2q^{13/2}x^{15/16+\varepsilon}$, we can extend the range of summation to remove the condition $S_i\le R_i$.

We're almost done. We use Lemma 2.3.2, and we may consider the integrals being for $V_i \ll x$, (else $S(\delta_1, \delta_2; V_1, V_2)$ vanishes). The error terms in (12) contribute for

$$\ll x^{1+\varepsilon} q^{-1} \tag{13}$$

to $S(d)$ if $d \leq q^{1/3}$.

Moreover, we can show that the main terms contribute to $S(d)$ a quantity

$$M \int_x^{2x} S(d;t) dt$$

where

$$S(d;t) = \sum_{\substack{r_i \leq l_i(t) \\ l=d_1 d_2}} \frac{\chi(r_1 r_2)}{d r_1 r_2} \log \frac{l_1(t)}{r_1} \log \frac{l_2(t)}{r_2} M(r_1) M(r_2) (d_1, r_1)(d_2, r_2) \tag{14}$$

is subject to

$$(r_i d_i, \alpha) = (r_1, r_2) = (r_1, d_2) = (r_2, d_1) = 1$$

We have defined

$$M(r) := \sum_{\substack{m|Q, m^2|r \\ m<q}} \mu(m)$$

If we denote by $S'(d;t)$ the expression resulting when we replace $M(r_i)$ by $N(r_i)$ (where $N(r) = 0$ if $p^2|r$ for some $\chi(p) = -1$ and $N(r) = 1$ else), then

$$S(d;t) - S'(d;t) \ll L^4 z^{-1} d^{-1} 4^{\omega(d)} \tag{15}$$

We now wish to extend (14) to all r_i satisfying

$$(r_i d_i, \alpha) = (r_1, r_2) = (r_1, d_2) = (r_2, d_1) = 1$$

We define, for $\sigma > 1$:

$$S'(d_i; t, \sigma) := \sum_{r_i \leq l_i(t)} \frac{\chi(r_1 r_2)}{(r_1 r_2)^\sigma} \log \frac{l_1(t)}{r_1} \log \frac{l_2(t)}{r_2} N(r_1 r_2)(d_1, r_1)(d_2, r_2)$$

$$S'(d_i, l_i, \sigma) := \sum_{r_i=1}^{\infty} \frac{\chi(r_1 r_2)}{(r_1 r_2)^\sigma} \log \frac{l_1}{r_1} \log \frac{l_2}{r_2} N(r_1 r_2)(d_1, r_1)(d_2, r_2)$$

where r_i is subject to the 4 conditions written just above. Then

$$S'(d_i; t, \sigma) - S(d_i, l_i(t), \sigma) \ll q d x^{\varepsilon - 1/2}$$

This error term together with (13) and (15) are small enough for Proposition 2.3.1.

It remains to evaluate the leading term. Observe that

$$S(d_i, l_i, \sigma) = f_{uv}(0,0)$$

where

$$f_{uv} = \partial^2 f(u,v)/\partial u \partial v, \;\; f(u,v) = l_1^u l_2^v \sum_{r_i} \chi(r_1 r_2) r_1^{-\sigma-u} r_2^{\sigma-v} N(r_1 r_2)(d_1, r_1)(d_2, r_2)$$

With the factorisation of $f(u,v)$ as an Euler product, we get

$$\sum_{d=d_1 d_2} f(u,v) = l_1^u l_2^v F(u,v) G(u,v)$$

where

$$F(u,v) = \prod_{\substack{p\nmid\alpha \\ \chi(p)=1}} \frac{1-p^{-2\sigma-u-v}}{(1-p^{-u-\sigma})(1-p^{-\sigma-v})} \prod_{\substack{p\nmid\alpha \\ \chi(p)=-1}} (1-p^{-\sigma-u}-p^{-\sigma-v})$$

$$G(u,v) = \prod_{p|d} \left(\left(2 + \frac{p}{p^{\sigma+u}-1} + \frac{p}{p^{\sigma+v}} \right) \frac{(1-p^{-u-\sigma})(1-p^{-v-\sigma})}{1-p^{-2\sigma-u-v}} \right)$$

Now, and until the end of the section, K_i denotes a continuous function of $\sigma \geq 1$, depending on i, q, χ, t and α but not on d. We also have $K_i \ll 1$ uniformly in q, t and σ (but not necessarily in α). We find that

$$\frac{F_u(0,0)}{F(0,0)} = \frac{L'(\sigma,\chi)}{L(\sigma,\chi)} + K_1$$

$$\frac{F_v(0,0)}{F(0,0)} = \frac{L'(\sigma,\chi)}{L(\sigma,\chi)} + K_1$$

$$F_{uv}(0,0) = F(0,0) \left(\left(\frac{L'(\sigma,\chi)}{L(\sigma,\chi)} \right)^2 + 2K_1 \frac{L'(\sigma,\chi)}{L(\sigma,\chi)} + K_2 \right)$$

We also have

$$G_u(0,0) = G_v(0,0) = G(0,0)A_1(d)$$

$$G_{uv}(0,0) = G(0,0)(A_2(d) + A_1(d)^2)$$

where A_i are additive functions of d, continuous functions of $\sigma \geq 1$, independent of t and uniformly in $\sigma \geq 1$ satisfying $A_1(p) \ll \log p,\ A_2(p) \ll (\log p)^2$.

Thus

$$\sum_{d=d_1d_2} S(d_i, l_i(t), \sigma) = F(0,0)G(0,0) \left(\left(\frac{L'(\sigma,\chi)}{L(\sigma,\chi)} \right)^2 + A_1(d)^2 + A_3(d) + L^2K_3 + LK_4 \frac{L'(\sigma,\chi)}{L(\sigma,\chi)} \right)$$

with A_3 as before but depends on t and satisfies

$$A_3(p) \ll \left(L + \left| \frac{L'(\sigma,\chi)}{L(\sigma,\chi)} \right| \right) \log p$$

Taking $\sigma \to 1$ we have $MF(0,0) \to \kappa x^{-1}$, $G(0,0) \to G(d)$. Proposition 2.3.1 follows by integrating over t since the integral of A_3 is also an additive function of d.

2.4 Proof of the statement

Let's add a final lemma. With the notations of Proposition 2.3.1, we have

Lemma 2.4.1. *Let* $\rho_1(d) = G(d)/d,\ \rho_2(d) = G(d)A'(d)/d$ *and* $\rho_3(d) = G(d)A(d)^2/d$. *Write* $p(\delta)$ *for the smallest prime factor of* δ *and write*

$$P(\delta) = \prod_{\substack{p|P \\ p<p(\delta)}} p$$

Set

$$S^{(i)}(\delta) = \sum_{d|P(\delta)} \rho_i(d\delta)\mu(d),\ S_i = \sum_{d|P} \rho_i(d)\mu(d)$$

Then $S^{(1)}(\delta), S_1 \geq 0$ *and for any* $\delta \leq q, \delta|P$, *we have*

$$S^{(2)}(\delta), S^{(3)}(\delta) \ll BLS^{(1)}(\delta)$$

Moreover

$$S_2, S_3 \ll BLS_1$$

Now that we have this one, we don't have to do a lot, it's mostly putting the pieces of the puzzle in the right order. We substitute S in (4), thanks to Proposition 2.3.1. We take $z \leq q^{1/3}, D = q^{1/3}$. The error terms then contribute to

$$\ll xL^4 z^{-1} \sum_{d \leq D} d^{-1} 4^{\omega(d)} \ll xL^8 z^{-1}$$

We'll now only turn to the upper bound of (4), for the lower bound can be studied in the very same way. To lighten the notations, we'll write λ_d instead of $\lambda_d^+(D)$. By [H.I80], there exists a set S depending on D such that if $\delta \in S$ and $\delta | P$, then $1 < \delta \leq \max(D, z^2) \leq q$. Moreover, for any function ρ we have

$$\sum_{d|P} \rho(d)\lambda_d = \sum_{d|P} \rho(d)\mu(d) + \sum_{\delta \in S, \delta | P} \sum_{d | P(\delta)} \rho(d\delta)\mu(d)$$

To obtain the bound $\delta \leq q$, one need the fact that the sieving limit satisfies $\beta \geq 3$ for the case of dimension 4. For convenience, we also write $G(d) = 0$ when $(d, \alpha) \neq 1$. Thanks to Lemma 2.4.1, if we write

$$S_i' := \sum_{d|P} \rho_i(d)\lambda_d$$

then

$$|S_i' - S_i| \ll BL|S_1' - S_1| \ (i = 2, 3)$$

From (4), the estimation of the contribution of the error terms and from what is right above, Proposition 2.3.1 leads to

$$S^{(3)} \leq \kappa S_1 \left\{ \left(\frac{L'(1,\chi)}{L(1,\chi)} \right)^2 + O(BL) \right\} + O(\kappa B^2 |S_1' - S_1|) + O(xL^8 z^{-1})$$

However, since $\log D / \log z = z_0/3$, [H.I80] shows that

$$S_1' - S_1 \ll \exp(-z_0/4) S_1$$

As $G(p) \leq 4$, we need a sieve of dimension 4, which leads to

$$S^{(3)} \leq \kappa S_1 \left\{ \left(\frac{L'(1,\chi)}{L(1,\chi)} \right)^2 + O(BL) + O(B^2 \exp(-z_0/4)) \right\} + O(xL^8 z^{-1})$$

By Lemma 2.2.5, whenever $z_0 \leq A \log\log \eta$

$$S^{(3)} \leq (1 + O(\exp(-z_0/4)) x\mathfrak{S}C(\alpha) + O(xL^8 z^{-1}) = x\mathfrak{S}C(\alpha) + O(x \exp(z_0/4))$$

By studying the lower bound, we get that

$$S^{(3)} = x\mathfrak{S}C(\alpha) + O(x \exp(z_0/4))$$

Putting this with Proposition 1.3.4, we get

$$S^{(0)} = x\mathfrak{S}C(\alpha) + O(xz_0^{-1})$$

and Theorem 1.3.2 follows by taking $z_0 = A \log\log \eta$.

This being proved, let's now prove Theorem 1.3.3. We know that the number of $n \in [\![x, 2x]\!]$ such that l_1 or l_2 is of the form p^e, with $e \geq 2$, is $O(\sqrt{x})$. Moreover, we have $\lambda(l_i) = \log x + O(1)$ if l_i is prime. Hence if

$$N(x) = \#\{n \leq x : l_i \textit{ both prime}\}$$

then, by Theorem 1.3.2

$$N(2x) - N(x) = \mathfrak{S}C(\alpha)x(\log x)^{-2} + O(xL^{-2}(\log\log \eta)^{-1})$$

uniformly for $q^{250} \leq x \leq \frac{q^{500}}{2}$. Thus

$$N(2x) - N(x) = \mathfrak{S}C(\alpha)x(\log x)^{-2} + O(xL^{-2}(\log\log \eta)^{-1}) + O(q^{250})$$

uniformly for $2x \leq q^{500}$. If we take X in the range $q^{300} \leq X \leq q^{500}$ and sum for $x = X/2, X/4, X/8, \ldots$, we get

Corollary 2.4.2.

$$\#\{n \leq x : l_1, l_2 \text{ both prime}\} = \mathfrak{S}C(\alpha)x(\log x)^{-2} + O(x(\log x)^{-2}(\log\log\eta)^{-1})$$

uniformly for $q^{300} \leq x \leq q^{500}$*. The implied constant is effective and depends only on* α_i, β_i*.*

If there exists a sequence of (q, χ, β_0) *for which* $\eta \to \infty$*, then one obtains*

$$\#\{n \leq x : l_1, l_2 \text{ both prime}\} \sim \mathfrak{S}C(\alpha)x(\log x)^{-2}$$

valid for each range $q^{300} \leq x \leq q^{500}$*.*

Now if we write A to be the implied constant, and if we take $C_3 = \exp\exp(2A(\mathfrak{S}C(\alpha))^{-1})$, then we get

Corollary 2.4.3. *If* $C_3 \leq \eta$ *for infinitely many triples* (q, χ, β_0)*, then there must be infinitely many* n *for which* l_1, l_2 *are simultaneously prime.*

This is what we meant by "in an appropriate sense" in Theorem 1.3.1. Now, to prove Theorem 1.3.3, we should consider two cases. In the first case, if there exists a sequence of triples (q, χ, β_0) such that $\eta \to \infty$, then take $C_1 = 1$. By Corollary 2.4.3, the first item of Theorem 1.3.3 holds.

In the second case, η is bounded (or zeros satisfying $\eta \geq 3$ -as we supposed- do not exist). So if $\eta \leq A$ for all β_0, χ, then take $C_1 = \min(A^{-1}, C_0)$. That ends the proof.

Now that the result is proved, one could get interested in the remark of Heath-Brown, who tried to get the reader interested in proving that if Siegel zeros exist, then there is always a prime number between x and $x + \sqrt{x}$, for x in a suitable interval (let's remind that neither Legendre's conjecture nor GRH are powerful enough to imply the existence of a prime between x and $x + \sqrt{x}$, for all x large enough). It actually has been proven by Friedlander and Iwaniec ([J.F04]) :

Theorem 2.4.4 (Friedlander, Iwaniec, 2004)**.** *Let* $\psi(x) := \sum_{n \leq x} \Lambda(n)$ *be the Chebychev summatory function. Let* χ *be the real primitive character of conductor* D*. Let* $x \geq D^r$ *with* $r = 18,290$ *and* $x^{39/79} \leq y \leq x$*. Then*

$$\psi(x) - \psi(x - y) = y\left(1 + O\left(L(1, \chi)(\log x)^{r^r}\right)\right)$$

The implied constant is absolute and computable.

This theorem is unconditional, but it only has a strong meaning if $L(1, \chi)$ is quite small (as mentioned in the section concerning the lacunarity). Friedlander and Iwaniec wrote that one may require

$$L(1, \chi) \ll (\log x)^{-r^r - 1}$$

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Appendix

Since this paper should be readable by a student in first year of master degree, a few things need to be defined/detailed.

First, the Dirichlet convolution, one very useful and common notion in arithmetic.

Definition 3.1.1. *Let f and g be two arithmetic functions (i.e whose domain is the positive integers and whose range is a subset of $\mathbb{C}$). We denote by $f \star g$ the Dirichlet convolution of f and g, an arithmetic function defined by*

$$(f \star g)(n) = \sum_{ab=n} f(a)g(b)$$

With this definition, we can define Möbius function, Liouville function and Van Mangold function.

1. μ is the unique arithmetic function such that $(1 \star \mu)(1) = 1$ and $(\mu \star 1)(n) = 0$ if $n > 1$. Another definition is given by

   $$\mu(n) := \begin{cases} 1, \ n = 1 \\ 0, \ \exists d > 1 : d^2 | n \\ (-1)^{\omega(n)} \text{ else} \end{cases}$$

   From this, we get the famous Möbius inversion formula : If $f = 1 \star g$, then $g = f \star \mu$.
2. λ, usually defined by $\lambda := (-1)^{\Omega}$. However, if χ_S is the indicator function of the set of perfect squares, then

   $$\lambda := \chi_S \star \mu$$
3. Last but not least, Λ, the Van Mangold function. We can define it either by

   $$\Lambda := \mu \star \log$$

   or by

   $$\Lambda(n) := \begin{cases} \log(p), \ \ n = p^e, \ p \text{ prime}, \ e \geq 1 \\ 0 \text{ else} \end{cases}$$

Another thing that one will often see in this paper : the Vinogradov asymptotic notation.

Definition 3.1.2. *There is actually not much to define, for the symbol only stands for the O notation. We write*

$$f \ll g$$

if and only if $f = O(g)$. Moreover, we write

$$f \asymp g$$

if and only if $f \ll g$ and $g \ll f$.

Let's now turn to more "complex" notions. The next definitions might have been introduced in the number theory class of master degree, but it doesn't take a lot of time to write it again.

Definition 3.1.3. *We say that a function χ from the integers to the complex numbers is a Dirichlet character if :*

1. $\exists k > 0 : \forall n, \chi(n) = \chi(k + n)$
2. $\chi(n) = 0 \Leftrightarrow (n, k) > 1$
3. $\chi(nm) = \chi(n)\chi(m)$

We say that χ is a Dirichilet character modulo k. To that character, we associate a function $L(\cdot, \chi) : s \mapsto \sum_{n \geq 1} \frac{\chi(n)}{n^s}$. It is said to be the Dirichlet L-function associated with the Dirichlet character χ.

We say that the modulus k is an induced modulus for χ if $\chi(a) = \chi(b)$ whenever a, b are congruent modulo k and each coprime to n. A character is primitive if there is no smaller induced modulus. In this case, k is called the conductor of χ.

Finally, a character is said principal if its only non zero value is 1. In this paper, all character are considered to be non-principal.

Here is a very useful and usual technique is to transform a sum into an integral, with the formula of *partial summation.*

Theorem 3.1.4 (Partial summation)**.** *Let a be an arithmetic function, $0 < y < x$ real numbers and $f \in \mathscr{C}^1([y,x];\mathbb{C})$. Let A be the summatory function of a. Then*

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt$$

Proof (Theorem 3.1.4)**.** *Let $\mathbb{1}(n,t) = 1$ if $n \leq t$ and 0 otherwise.*

$$\begin{aligned}\int_y^x A(t)f'(t)dt &= \int_y^x \sum_{n\leq x} a(n)\mathbb{1}(n,t)f'(t)dt \\ &= \sum_{n\leq x} a(n)\int_y^x \mathbb{1}(n,t)f'(t)dt \\ &= \sum_{n\leq x} a(n)\int_{\max(n,y)}^x f'(t)dt \\ &= \sum_{n\leq x} a(n)f(x) - \sum_{n\leq y} a(n)f(y) - \sum_{y<n\leq x} a(n)f(n)\end{aligned}$$

And the result follows.

Another common trick to tackle certain sum of convoluted arithmetic functions is the Dirichlet hyperbola method.

Theorem 3.1.5 (Dirichlet hyperbola method)**.** *Let f, g be two arithmetic functions with respective summatory functions F, G. Then for all $1 \leq y \leq x$*

$$\sum_{n\leq x}(f\star g)(n) = \sum_{n\leq y} g(n)F(x/n) + \sum_{n\leq x/y} f(n)G(x/n) - F(x/y)G(y)$$

Proof (Theorem 3.1.5)**.**

$$\begin{aligned}\sum_{n\leq x}(f\star g)(n) = \sum_{mn\leq x} f(m)g(n) &= \sum_{mn\leq x, n\leq y} f(m)g(n) + \sum_{mn\leq x, n>y} f(m)g(n) \\ &= \sum_{n\leq y} g(n)F(x/n) + \sum_{m\leq x/y} f(m)(G(x/m) - G(y)) \\ &= \sum_{n\leq y} g(n)F(x/n) + \sum_{m\leq x/y} f(m)G(x/m) - G(y)F(x/y)\end{aligned}$$

Finally, we formally state the two conjectures mentioned in the paper. First, Legendre's conjecture, the natural step after Bertrand's postulate (which states that there is a prime between $n+1$ and $2n-3$, if $n \geq 3$).

Conjecture (Legendre, ∼1810)**.**

$$\forall n \geq 1, \exists p : n^2 \leq p \leq (n+1)^2$$

Finally, we'll end on what probably is the most famous statement in the history of mathematics, the Generalized Riemann Hypothesis.

Conjecture (Piltz, 1884)**.** *Let χ be a Dirichlet character, and let $s \in \mathbb{C}$. If s is not a negative real number and if $L(s,\chi) = 0$, then $\Re(s) = 1/2$.*